FINITE DIMENSIONAL REPRESENTATIONS OF THE SOFT TORUS

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Abstract. The soft tori constitute a continuous deformation, in a very precise sense, from the commutative C^*-algebra $C(\mathbb{T}^2)$ to the highly non-commutative C^*-algebra $C^*(\mathbb{F}_2)$. Since both of these C^*-algebras are known to have a separating family of finite dimensional representations, it is natural to ask whether that is also the case for the soft tori. We show that this is in fact the case.

1. Introduction

Knowing that a given C^*-algebra has many representations on finite dimensional Hilbert spaces is of great importance to understanding structural properties of it. Among C^*-algebras, those who possess a separating family of finite dimensional representations are called *residually finite dimensional* or just *RFD*. This class was studied in [12], [11] and [1], and more recent insight about it has lead to important advances in classification theory and the theory of quasidiagonal C^*-algebras (see, e.g., [2], [5] and [6]).

For any $\varepsilon \geq 0$ we define a C^*-algebra A_ε as the unital universal C^*-algebra defined by the generators u, v subject to the relations

$$uu^* = u^*u = 1 \qquad vv^* = v^*v = 1 \qquad \|uv - vu\| \leq \varepsilon.$$

As recorded in [8], A_0 is the commutative C^*-algebra of functions over the torus $\mathbb{T}^2$, and A_ε is the full C^*-algebra of the free group of two generators $\mathbb{F}_2$ whenever $\varepsilon \geq 2$. For ε between 0 and 2 we get a class of C^*-algebras which are commonly referred to as *soft tori*. These C^*-algebras are of relevance to several problems in operator algebra theory (see [10]) and have been extensively studied in [3], [8], [9], [7].

The starting point of the investigation reported on in the present paper is a result from [9], stating that the soft tori form a continuous field interpolating between the (hard) torus and the group C^*-algebra

Date: February 1, 2008.

1991 *Mathematics Subject Classification.* Primary 46L05; secondary 46L85, 47B20.

Partially supported by the Carlsberg Foundation.

of the free group. Since $C(\mathbb{T}^2)$ is obviously RFD, and since $C^*(\mathbb{F}_2)$ was proved to be RFD in [4] — a surprise at the time — we are naturally led to the question of whether the same is true for the interpolating family A_ε. We are going to prove that this is the case.

2. Methods

We are going to employ a method from [3] and [8] which lies behind many results about the structural properties of A_ε.

We define B_ε as the universal C^*-algebra given by the generators $\{u_n\}_{n\in\mathbb{Z}}$ and the relations

$$u_n u_n^* = u_n^* u_n = 1 \qquad \|u_{n+1} - u_n\| \leq \varepsilon. \tag{1}$$

Clearly one can define an automorphism α on B_ε by

$$u_n \mapsto u_{n+1},$$

and as seen in [8] one has

$$A_\varepsilon = B_\varepsilon \rtimes_\alpha \mathbb{Z}.$$

There is a faithful conditional expectation $E_\alpha : A_\varepsilon \longrightarrow B_\varepsilon$.

Our strategy is to prove that A_ε is RFD by proving that B_ε is RFD in a way which is covariant with α.

3. Finite dimensional representations of B_ε

We start out by finding a new picture of B_ε by generators and relations.

Lemma 3.1. *For any $\varepsilon < 2$, B_ε is isomorphic to the universal C^*-algebra generated by $v_0, \{h_n\}_{n\in\mathbb{Z}}$ subject to the relations*

$$v_0 v_0^* = v_0^* v_0 = 1 \qquad h_n = h_n^* \qquad \|h_n\| \leq 2\cos(\varepsilon/2) \tag{2}$$

Proof. Let us denote the C^*-algebra generated by v_0 and h_n subject to (2) by B'_ε. We can define a map $\varphi : B_\varepsilon \longrightarrow B'_\varepsilon$ by

$$u_n \mapsto \begin{cases} e^{i\pi h_n} \cdots e^{i\pi h_1} v_0 & n > 0 \\ v_0 & n = 0 \\ e^{-i\pi h_n} \cdots e^{-i\pi h_{-1}} v_0 & n < 0 \end{cases}$$

since the elements to the right of the arrow above satisfy the relations (1). Similarly, the universal property of B'_ε allows for a map $\psi : B'_\varepsilon \longrightarrow B_\varepsilon$ defined by

$$v_0 \mapsto u_0 \qquad h_n \mapsto \frac{1}{i\pi} \mathrm{Log}(u_{n+1} u_n^*)$$

Clearly φ and ψ are each others' inverse. $\square$

This characterization can be used to shorten the proof of [8, 2.2], stating that B_ε is homotopic to $C(\mathbb{T})$. To see this, define maps $\varphi : B'_\varepsilon \longrightarrow C(\mathbb{T})$ and $\psi : C(\mathbb{T}) \longrightarrow B'_\varepsilon$ by the correspondence $v_0 \leftrightarrow [z \mapsto z]$, $h_n \leftrightarrow 0$. Clearly $\varphi\psi = \mathrm{id}_{C(\mathbb{T})}$, and $\chi_t : B'_\varepsilon \longrightarrow B'_\varepsilon$ given by

$$\chi_t(v_0) = v_0 \qquad \chi_t(h_n) = th_n$$

provides a homotopy from $\mathrm{id}_{B'_\varepsilon}$ to $\psi\varphi$.

In the following proof, we denote by $\mathrm{Alg}(X)$ the smallest $*$-algebra, not necessarily closed, generated by X inside some C^*-algebra.

Proposition 3.2. *For any $\varepsilon < 2$, B_ε is RFD. In fact, for any $0 \neq b \in B_\varepsilon$ there exists $n \in \mathbb{N}$, an automorphism β of $\mathbf{M}_n$ and a representation $\rho : B_\varepsilon \longrightarrow \mathbf{M}_n$ with the properties:*

$$\rho(b) \neq 0 \qquad \beta\rho = \rho\alpha \qquad \beta^n = \mathrm{id}\,.$$

Proof. For the first claim we use the characterization of B_ε given by the relations (2) and proceed as in [4]. Fix a faithful non-degenerate representation $\pi : B_\varepsilon \longrightarrow \mathbb{B}(\mathcal{H})$, where we may assume that $\mathcal{H}$ is a separable Hilbert space.

Let P_m be a sequence of projections, with $\mathrm{rank}(P_m) = m$, converging strongly to the unit of $\mathbb{B}(\mathcal{H})$, and abbreviate

$$T_{0,m} = P_m\pi(v_0)P_m \qquad K_{n,m} = P_m\pi(h_n)P_m$$

Now note that for each m the collection of elements $\{V_{0,m}, H_{n,m} : n \in \mathbb{Z}\}$ defined by

$$V_{0,m} = \begin{bmatrix} T_{0,m} & \sqrt{P_m - T_{0,m}T_{0,m}{}^*} \\ \sqrt{P_m - T_{0,m}{}^*T_{0,m}} & T_{0,m}{}^* \end{bmatrix} \qquad H_{n,m} = \begin{bmatrix} K_{n,m} & 0 \\ 0 & K_{n,m} \end{bmatrix}$$

satisfies (2) in $\mathbf{M}_2(P_m\mathbb{B}(\mathcal{H})P_m) \simeq \mathbf{M}_{2m}$. Consequently we get representations $\pi_m : B_\varepsilon \longrightarrow \mathbf{M}_{2m}$. We are going to check, following [4], that

$$\underline{\pi} : B_\varepsilon \longrightarrow \prod_{m=1}^{\infty} \mathbf{M}_{2m} \qquad \underline{\pi}(b) = (\pi_m(b))_{m=1}^{\infty}$$

is an isometry. It suffices to check that $\|\underline{\pi}(x)\| \geq \|x\| - \eta$ for any $\eta > 0$ and any $x \in \mathrm{Alg}(\{v_0, h_{-N}, \ldots, h_N\})$. Fix η, N and x and write

$$x = F(v_0, h_{-N}, \ldots, h_N)$$

where F is some finite linear combination of finite words in $2N + 2$ variables and their adjoints. Since

$$V_{0,m} \longrightarrow \begin{bmatrix} \pi(v_0) & 0 \\ 0 & \pi(v_0)^* \end{bmatrix} \qquad H_{n,m} \longrightarrow \begin{bmatrix} \pi(h_n) & 0 \\ 0 & \pi(h_n) \end{bmatrix}, \qquad m \longrightarrow \infty$$

strongly on the unit ball of $\mathbf{M}_2(\mathbb{B}(\mathcal{H}))$ we conclude that

$$\begin{aligned}\lim_{m\longrightarrow\infty} &\|F(V_{0,m}, H_{-N,m}, \ldots, H_{N,m})\| \\ &= \left\| \begin{bmatrix} \pi(F(v_0, h_{-N}, \ldots, h_N)) & 0 \\ 0 & \pi(F(v_0^*, h_{-N}, \ldots, h_N)) \end{bmatrix} \right\| \\ &\geq \|\pi(x)\| = \|x\|.\end{aligned}$$

We can hence find m such that

$$\|\underline{\pi}(x)\| \geq \|\pi_m(x)\| = \|F(V_{0,m}, H_{-N,m}, \ldots, H_{N,m})\| \geq \|x\| - \eta.$$

For the second claim, we go back to the original presentation (1) of B_ε by unitary generators only. For a given $b \in B_\varepsilon$ with $\|b\| = 1$ we fix, using the first part of the proposition, a finite dimensional representation $\pi : B_\varepsilon \longrightarrow \mathbf{M}_m$ with $\|\pi(b)\| > \frac{3}{4}$. We also fix c and $N \in \mathbb{N}$ such that

$$\|b - c\| < \tfrac{1}{4} \qquad c \in \mathrm{Alg}(u_{-N}, \ldots, u_N).$$

Choose $M > 0$ and unitaries $v_0^\pm, \ldots, v_M^\pm$ with the properties

$$\left\|v_{n+1}^\pm - v_n^\pm\right\| \leq \varepsilon \qquad v_0^\pm = u_{\pm N} \qquad v_M^\pm = 1.$$

There is then exactly one representation $\pi' : B_\varepsilon \longrightarrow \mathbf{M}_m$ which is $2(N+M)$-periodic in the sense that $\pi'(u_n) = \pi'(u_{n+2(N+M)})$ and satisfies

$$\pi'(u_n) = \begin{cases} v_{-N-n}^- & -M - N \leq n < -N \\ \pi(u_n) & -N \leq n \leq N \\ v_{n-N}^+ & N < n \leq N + M \end{cases}$$

Note that $\pi'(c) = \pi(c)$; in particular $\|\pi'(c)\| \geq \frac{1}{2}$.

Now let $n = 2(N + M)m$. With β defined as the forward cyclic shift in block form (with period $2(N + M)$) we may define a covariant representation ρ of B_ε on $\mathbf{M}_n$ by

$$u_i \mapsto \begin{bmatrix} \pi'(u_i) & & & \\ & \pi'(u_{i+1}) & & \\ & & \ddots & \\ & & & \pi'(u_{i+2(N+M)-1}) \end{bmatrix}.$$

We have

$$\|\rho(b)\| \geq \|\pi'(b)\| \geq \|\pi'(c)\| - \tfrac{1}{4} > 0$$

□

4. Finite dimensional representations of A_ε

Theorem 4.1. *For any $\varepsilon > 0$, let A_ε be the universal C^*-algebra generated by a pair of unitaries subject to the relation $\|uv - vu\| \leq \varepsilon$. Then A_ε is residually finite dimensional in the sense that it admits a separating family of finite dimensional representations.*

Proof. We may assume that $0 < \varepsilon < 2$. Let $0 \neq a \in A_\varepsilon$. Then also $b = E_\alpha(a^*a)$ is nonzero, for the conditional expectation is faithful. Choose n, ρ and β as in Proposition 3.2 and define

$$\pi : A_\varepsilon = B_\varepsilon \rtimes_\alpha \mathbb{Z} \longrightarrow \mathbf{M}_n \rtimes_\beta \mathbb{Z}$$

as the extension to the crossed product of the covariant $*$-homomorphism ρ. We then have, with E_β the conditional expectation from $\mathbf{M}_n \rtimes_\beta \mathbb{Z}$ to $\mathbf{M}_n$

$$E_\beta(\pi(a^*a)) = \pi(E_\alpha(a^*a)) = \rho(b) \neq 0,$$

so $\pi(a) \neq 0$.

Note finally that since β is periodic, any irreducible representation of $\mathbf{M}_n \rtimes_\beta \mathbb{Z}$ is finite dimensional, and hence this C^*-algebra is RFD. Therefore we may compose π with some finite dimensional representation of it to exhibit a finite dimensional representation which does not vanish on a. □

Linear algebra tells us that $\mathbf{M}_n$ has a faithful tracial state and has the property that every matrix which is *hyponormal* is the sense that

$$x^*x \geq xx^*$$

is in fact normal. As in [4], we may conclude:

Corollary 4.2. *For any ε, A_ε has a faithful tracial state. Furthermore, any hyponormal operator in A_ε is in fact normal.*

Proof. These two properties clearly pass from matrices to sums of the form $\bigoplus_{n\in\mathbb{N}} \mathbf{M}_{m_n}$, and from these sums to any of their subalgebras. By the theorem, A_ε is one such. □

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